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FAST TRACK COMMUNICATION

Quantum hydrodynamics and nonlinear differential equations for degenerate Fermi gas

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Abstract
We present new nonlinear differential equations for spacetime correlation functions of Fermi gas in one spatial dimension. The correlation functions we consider describe non-stationary processes out of equilibrium. The equations we obtain are integrable equations. They generalize known nonlinear differential equations for correlation functions at equilibrium [1–4] and provide vital tools for studying non-equilibrium dynamics of electronic systems. The method we developed is based only on Wick's theorem and the hydrodynamic description of the Fermi gas. Differential equations appear directly in bilinear form.

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1. Introduction

Various spacetime correlation functions of free Fermi gas in one spatial dimension (and models related to it—impenetrable Bose and Fermi gases, XY-spin chain, 2D Ising model, etc) are Fredholm determinants or their minors and, therefore, generally speaking, obey integrable nonlinear differential equations with respect to spacetime and other parameters such as temperature, chemical potential, etc. In practice, a derivation of these equations is a complicated task. Few equations are known. However, once obtained, they are indispensable in studies of off-shell properties of Fermi gas.

Historically nonlinear differential equations for correlation functions first appeared in studies of the 2D Ising model [1]. In [2] these equations have been extended, applied to the XY-spin chain, and have been derived in explicitly bilinear form with the help of the

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Wick theorem generalized for these purposes in [5]. In [3] it has been shown that at zero temperature the two-point, equal-time correlation function of impenetrable bosons and of the XY-spin chain can be expressed through Painlevé transcendents. These equations have been extended to time- and temperature-dependent correlation functions at equilibrium for impenetrable bosons and the XY-chain in [4, 6].

The derivation of the known equations was essentially based on equilibrium and translational invariance properties of correlation functions. Although they have been recognized as integrable, their relations to integrable hierarchies remained unclear.

Recently, interest focused on electronic systems out of equilibrium [7] and especially on non-stationary properties of propagating localized states. There the physical properties change drastically and the dynamics acquires essentially nonlinear features such as hydrodynamic instabilities [8]. Apart from being of a fundamental interest, research in non-equilibrium degenerate Fermi gas is driven by a quest to transmit quantum information in electronic nanodevices and to control and manipulate entangled quantum many-body states. Integrable equations describing the non-equilibrium—non-stationary states of electronic gas, once available, would be a valuable tool in studies of a complex dynamics of electronic gas. They are derived in this communication.

We will obtain nonlinear differential equations with respect to spacetime for the correlation functions of vertex operators

$$e^{\mathrm{i}a_L\varphi_L}\cdot e^{\mathrm{i}a_R\varphi_R}, \tag{1}$$

where $\varphi_{R,L}$ are the right/left handed chiral Bose fields of fermionic currents and $a_{L,R}$ are arbitrary (not necessarily real) parameters. The definition of these fields in terms of free fermions is given below. Equations with respect to parameters of the density matrix (e.g., temperature or chemical potential) can also be obtained, but are not discussed here.

The correlation functions of vertex operators (1) are important in many applications. In the case of equilibrium they also satisfy some previously known equations. A few particular cases are worth mentioning. The case $a_L = a_R = 1/2$ describes a one-point function of impenetrable bosons [3, 4]. The imaginary $a_L = a_R$ and $a_L = -a_R$ yield a generating function of moments of the distribution function of the density and velocity of electronic gas. The chiral correlation functions $a_L = 0$ or $a_R = 0$ describe a tunneling into an electron gas (fermi edge singularity) [9], etc.

In this communication, we give a simple *physical* derivation of equations on these correlation functions. Our derivation is based solely on a general form of the Wick theorem[4], the representation of Fermi-operators through Bose fields (see, e.g., [10]), and the hydrodynamic description of the Fermi gas [11]. Our equations appear directly in Hirota's bilinear form[5]. They are known as *modified Kadomtsev–Petviashvili* (or mKP) equations and are the first and the second equation of mKP hierarchy. In a particular case, in free fermion systems these equations have previously appeared in [8, 12]. The nonlinear equations being integrable, are proven to be an effective tool in computing asymptotes of correlation functions in different regimes. These computations are specific to each particular problem and will not be discussed here (see [8] for some practical applications).

In the following after necessary preliminaries on the Fermi–Bose correspondence, we describe the correlation functions (5), present the nonlinear differential equations they obey (16), (18) and (19), and then give a short derivation of those equations.

[4] For a general form of Wick's theorem see, e.g., [5].

[5] The equations for correlation functions of quantum systems in Hirota bilinear form first appeared in studies of XY-spin chain in [2].

2. Fermions and a Bose field

We consider free fermions on a circle of a circumference L,

$$H = \sum_p \frac{p^2}{2m} \psi_p^\dagger \psi_p, \tag{2}$$

where ψ_p is a mode of a fermion field $\psi(x) = \frac{1}{\sqrt{L}} \sum_p \mathrm{e}^{\frac{\mathrm{i}}{\hbar} px} \psi_p$.

A central point of our approach is the hydrodynamic description of the Fermi gas. We briefly review it. The properties of the Fermi gas are fully described in terms of canonical hydrodynamic variables, density and velocity,

$$\rho = \frac{1}{2\pi}(J^R + J^L), \qquad v = \frac{\hbar}{2m}(J^L - J^R),$$

where the chiral (right, left) currents are

$$J^{R,L}(x) = \frac{2\pi}{L} \sum_k \mathrm{e}^{\frac{\mathrm{i}}{\hbar} kx} J_k^{R,L}, \qquad J_k^{R,L} = \sum_{\pm p>0} \psi_p^\dagger \psi_{p+k}.$$

Tomonaga's equal time commutation relations of currents (see, e.g., [10])

$$\left[J_k^R, J_l^R\right] = \left[J_k^L, J_l^L\right] = \frac{Lk}{2\pi\hbar} \delta_{k+l,0}, \qquad \left[J_k^R, J_l^L\right] = 0 \tag{3}$$

lead to a canonical relation between density and velocity,

$$[\rho(x,t), v(y,t)] = -\mathrm{i}\frac{\hbar}{m} \nabla\delta(x-y), \tag{4}$$

and to the canonical Bose field:

$$\nabla\varphi_{R,L} \equiv \pm J^{R,L}(x,t) = \frac{m}{\hbar} v \pm \pi\rho, \qquad [\varphi_L, \varphi_R] = 0.$$

We will be interested in the spacetime dependence of correlations of two vertex operators:

$$\mathrm{tr}(:\mathrm{e}^{-\mathrm{i}b_L\varphi_L t(\xi') - ib_R\varphi_R(\xi')} :: \mathrm{e}^{\mathrm{i}a_L\varphi_L(\xi) + ia_R\varphi_R(\xi)} : \varrho). \tag{5}$$

Here ϱ is the density matrix and $\xi = (x,t)$. The colon denotes normal ordering with respect to the vacuum[6].

Below we drop the chiral subscripts R, L in all formulae and assume that the upper (lower) sign corresponds to the right (left) sectors.

3. Coherent states

We will consider correlation functions with respect to *coherent states*. This means that the density matrix is an element of $gl(\infty)$, i.e., it is an exponent bilinear in fermionic modes:

$$\varrho = \exp\left(\sum_{p,q} A_{pq} \psi_p^\dagger \psi_q\right).$$

This choice of the density matrix allows us to use Wick's theorem.

[6] The normal ordering of hydrodynamic modes reads that the negative modes of the right chiral current $J_k^R, k < 0$, and positive modes of the left chiral current $J_k^L, k > 0$ are placed to the left of the rest of the modes. For example $:\mathrm{e}^{\mathrm{i}a\varphi} := \mathrm{e}^{\mathrm{i}a\varphi_+} \mathrm{e}^{\mathrm{i}a\varphi_-}$, where $\varphi_-(x) = \mp\mathrm{i}\frac{2\pi\hbar}{L} \sum_{\pm k>0} \frac{1}{k} J_k \mathrm{e}^{\frac{\mathrm{i}}{\hbar}kx}$ and $\varphi_+(x) = \mp\mathrm{i}\frac{2\pi\hbar}{L} \sum_{\pm k<0} \frac{1}{k} J_k \mathrm{e}^{\frac{\mathrm{i}}{\hbar}kx}$.

The set of coherent states is rather general. It exhausts most interesting applications. For example, the density matrix can be a Boltzmann distribution, or any other non-equilibrium distribution as in [7]. Another choice would be $\varrho = |V\rangle\langle V|$, where $|V\rangle = \exp\left(\sum_k t_k J_k\right)|0\rangle$ is a coherent state (i.e., an eigenstate of current operators). Such states appear as a result of shake up of the electronic gas by electromagnetic potential with harmonics t_k.

4. Quantum hydrodynamics

The Hamiltonian of free fermions (2) in the sector of coherent states can be expressed solely in terms of density and velocity[7]:

$$H = \int \left(\frac{m{:}\rho v^2{:}}{2} + \frac{\pi^2\hbar^2}{6m}{:}\rho^3{:} \right) \mathrm{d}x. \tag{6}$$

The evolution of density and velocity follows

$$:\dot{\rho}: + \nabla : (\rho v): = 0 \qquad \text{continuity equation}$$

$$:\dot{v}: + \frac{1}{2}\nabla : \left(v^2 + \frac{\pi^2\hbar^2}{m^2}\rho^2 \right): = 0, \qquad \text{Euler's equation.}$$

The hydrodynamic equations are decoupled into two independent chiral quantum Riemann equations

$$\dot{\varphi} + \frac{\hbar}{2m}{:}(\nabla\varphi)^2{:} = 0. \tag{7}$$

5. Vertex operator and correlation functions

The Heisenberg evolution equations for the chiral vertex operators follow from the quantum Riemann equation (7) (see appendix A):

$$\left(\mathrm{i}\partial_t - \frac{\hbar}{2m}\nabla^2 \right){:}\mathrm{e}^{\mathrm{i}a\varphi}{:} = \frac{\hbar}{2m}a(a+1){:}\mathrm{e}^{\mathrm{i}a\varphi}T{:}, \tag{8}$$

$$\left(\mathrm{i}\partial_t + \frac{\hbar}{2m}\nabla^2 \right){:}\mathrm{e}^{\mathrm{i}(a+1)\varphi}{:} = -\frac{\hbar}{2m}a(a+1){:}\mathrm{e}^{\mathrm{i}(a+1)\varphi}\bar{T}{:}, \tag{9}$$

where $T = {:}(\nabla\varphi)^2{:} - \mathrm{i}\nabla^2\varphi$ and $\bar{T} = {:}(\nabla\varphi)^2{:} + \mathrm{i}\nabla^2\varphi$ are holomorphic (antiholomorphic) components of the stress–energy tensor of a chiral Bose field (with the central charge 1/2).

A particular consequence of these equations is that the vertex operators $\mathrm{e}^{\pm\mathrm{i}\varphi_{R,L}}$ obey the Schroedinger equation. This yields to a familiar formula representing fermions as exponents of a Bose field:

$${:}\mathrm{e}^{\mathrm{i}\varphi}{:} \simeq \sqrt{L}\psi(x), \qquad {:}\mathrm{e}^{-\mathrm{i}\varphi}{:} \simeq \sqrt{L}\psi^\dagger(x). \tag{10}$$

A useful representation for current $J = \pm{:}\mathrm{e}^{-\mathrm{i}\varphi}(-\mathrm{i}\nabla)\,\mathrm{e}^{\mathrm{i}\varphi}{:}$ and the stress–energy tensor reads

$$T = -{:}\mathrm{e}^{-\mathrm{i}\varphi}\nabla^2\,\mathrm{e}^{\mathrm{i}\varphi}{:}, \qquad \bar{T} = -{:}\mathrm{e}^{\mathrm{i}\varphi}\nabla^2\,\mathrm{e}^{-\mathrm{i}\varphi}{:}. \tag{11}$$

Current and stress–energy tensor can be further cast in fermionic form using (10) and OPE for vertex operators as follows from the current algebra (4):[8]

$${:}\mathrm{e}^{\mathrm{i}a\varphi(x)}\,\mathrm{e}^{-\mathrm{i}b\varphi(x')}{:} = \left(\frac{\mp\mathrm{i}2\pi(x-x')}{L} \right)^{ab} {:}\mathrm{e}^{\mathrm{i}a\varphi(x)}{:}{:}\mathrm{e}^{-\mathrm{i}b\varphi(x')}{:}. \tag{12}$$

[7] To the best of our knowledge the Hamiltonian for quantum hydrodynamics of free Fermi gas in the form equivalent to (6) appeared first in [11].

[8] In this and further formulae we assume a radial ordering $\pm\mathrm{Im}(x - x') > 0$ and $|x - x'| \ll L$.

With the help of (12), (11) and (10) we write

$$T = \mp 4\pi \mathrm{i}\psi^\dagger \nabla\psi, \qquad \bar{T} = \mp 4\pi \mathrm{i}\psi\nabla\psi^\dagger, \qquad J = 2\pi\psi^\dagger\psi. \tag{13}$$

Finally, we cast the equation of motion of the vertex operator in a suggestive bilinear form (quantum Hirota equation)

$$\left(\mathrm{i}D_t - \frac{\hbar}{2m}D_x^2\right) : \mathrm{e}^{\mathrm{i}a\varphi} \cdot \mathrm{e}^{\mathrm{i}(a+1)\varphi} : = 0. \tag{14}$$

Here D is the Hirota derivative $Df \cdot g = \partial f g - f\partial g$ and $D^2 f \cdot g = \partial^2 f g - 2\partial f \partial g + f\partial^2 g$. This equation follows from the quantum Riemann equation (7).

6. Nonlinear differential equations

The nonlinear equations themselves do not depend on the parameters $a_{L,R}, b_{L,R}$ in (5). For brevity we put $L = 2\pi$ in formulae below and consider only the chiral and neutral ($a = b$) correlation function, which we denote as

$$\tau_a(\xi, \xi') = \mathrm{tr}(: \mathrm{e}^{-\mathrm{i}a\varphi(\xi')} :: \mathrm{e}^{\mathrm{i}a\varphi(\xi)} : \varrho). \tag{15}$$

Being presented in a bilinear form the differential equations for correlation functions look identical to the quantum equation (14):

$$\left(\mathrm{i}D_t - \frac{\hbar}{2m}D_x^2\right) \tau_a \cdot \tau_{a+1} = 0. \tag{16}$$

The equations must be viewed as equations for $\tau_{a+n}(x, t; x', t')$ in the continuous variables x, t and the discrete variable n, where $|a| \leqslant 1/2$. As such they are a closed set of equations known as the mKP equations. The correlation function can be found given initial conditions at $t = 0$, for all integer n and any x. Certain asymptotes or analytical conditions may also be helpful in solving the problem for particular cases. This program is taken up in practice in [8], where in [8] the density matrix had the form $|A\rangle\langle 0|$ where $|0\rangle$ is the ground state of electron gas and $|A\rangle$ is a chiral coherent state of the form $|A\rangle = \varrho|0\rangle$. We also comment that τ_a where $|a| \leqslant 1/2$ is a Fredholm determinant, while τ_{a+n} are its minors.

The mKP equations (16) have many different forms discussed in the literature. We will mention one of them. Setting

$$\Phi = \mathrm{i}\frac{\hbar}{m}\log\left(\frac{\tau_a}{\tau_{a+1}}\right), \qquad \tilde{\Phi} = \frac{\hbar}{m}\log(\tau_a\tau_{a+1})$$

equation (16) becomes

$$\partial_t \Phi = -\frac{1}{2}(\nabla\Phi)^2 + \frac{\hbar}{2m}\nabla^2\tilde{\Phi}. \tag{17}$$

In this form the classical equation (17) resembles the quantum equation (7). The last term in (17) can be considered as a quantum correction to the semiclassical Riemann equations.

In addition to the evolution (16) there is another equation which does not involve time evolution. This equation connects correlation functions with different density matrices: ϱ and $\tilde{\varrho} = \sum_{pq} a_{pq}\psi_p^\dagger\varrho\psi_q$, where a_{pq} is an arbitrary non-degenerate matrix. Then τ_a and $\tilde{\tau}_a = \mathrm{tr}(:\mathrm{e}^{-\mathrm{i}a\varphi(\xi')}::\mathrm{e}^{\mathrm{i}a\varphi(\xi)}:\tilde{\varrho})$ are related by

$$D_x D_{x'}\tilde{\tau}_a \cdot \tau_a = a^2(\tilde{\tau}_{a+1}\tau_{a-1} + \tau_{a+1}\tilde{\tau}_{a-1}). \tag{18}$$

This equation is a minor generalization of the familiar 2D Toda lattice equation.

An important particular case occurs when the modes p, q in $\tilde{\varrho}$ lie very far from the Fermi surface. Then $\tilde{\tau}_a \sim \tau_a$ and equation (18) becomes the 1D Toda chain equation

$$D_x D_{x'} \tau_a \cdot \tau_a = 2a^2 \tau_{a+1} \tau_{a-1}. \tag{19}$$

This equation, together with (16) being written for pairs τ_a, τ_{a+1} and τ_{a-1}, τ_a give a closed set which involves only $\tau_{a-1}, \tau_a, \tau_{a+1}$.

7. Translation invariant states

Further closure can be achieved for special cases. For example, if the density matrix commutes with momentum, the correlation function depends on the difference $x - x'$ (e.g., the density matrix depends only on occupation numbers $n_p = \psi_p^\dagger \psi_p, \varrho = \exp \sum_p \lambda_p n_p$. Then it also commutes with the Hamiltonian, is a function of $t - t'$, and describes stationary processes). In this case, we denote the tau function as

$$\tau_a(x, t; x', t') = F_a(x - x', t, t').$$

As follows from (16), (19) three functions F_{a-1}, F_a, F_{a+1} obey three equations with respect to (x, t):

$$\begin{aligned} &\left(\mathrm{i}D_t - \frac{\hbar}{2m} D_x^2\right) F_a \cdot F_{a+1} = 0, \\ &\left(\mathrm{i}D_t + \frac{\hbar}{2m} D_x^2\right) F_a \cdot F_{a-1} = 0, \\ &D_x^2 F_a \cdot F_a = -2a^2 F_{a-1} F_{a+1}. \end{aligned} \tag{20}$$

Here we used $D_{x'} = -D_x$.

These equations are the bilinear form of the nonlinear Schroedinger equation (without a complex involution). Introducing $\Psi = F_{a+1}/F_a$ and $\tilde{\Psi} = F_{a-1}/F_a$ we have

$$\begin{aligned} &\left(\mathrm{i}\partial_t + \frac{\hbar}{2m} \nabla^2\right) \Psi = \frac{\hbar}{m} a^2 \tilde{\Psi} \Psi^2, \\ &\left(-\mathrm{i}\partial_t + \frac{\hbar}{2m} \nabla^2\right) \tilde{\Psi} = \frac{\hbar}{m} a^2 \Psi \tilde{\Psi}^2. \end{aligned} \tag{21}$$

These equations were obtained for time- and temperature-dependent two-point correlation functions for impenetrable bosons at equilibrium ($\varrho = \mathrm{e}^{-H/T}$) [4]. The impenetrable bosons are equivalent to free fermions. The creation operator of the boson is the vertex operator (1) with $a_L = a_R = 1/2$.

8. Derivation of nonlinear equations

The fact that the quantum equation for vertex operators (14) and the classical equations for the correlation functions (16) have the same form may not be an accident but may reflect a general property of integrable equations written in the bilinear form. Here we limit our discussion by demonstrating this phenomena for equation (16).

We now prove the main formula (16). First, we adopt the notation

$$\begin{aligned} \langle \mathcal{O}_1(x), \mathcal{O}_2(x') \rangle &= (\tau_a(x, t; x', t'))^{-1} \\ &\quad \times \operatorname{tr}(\varrho\, \mathrm{e}^{\mathrm{i}Ht} {:}\mathrm{e}^{\mathrm{i}a\varphi(x)} \mathcal{O}_1(x){:} \mathrm{e}^{\mathrm{i}H(t-t')} {:}\mathrm{e}^{-\mathrm{i}a\varphi(x')} \mathcal{O}_2(x'){:} \mathrm{e}^{-\mathrm{i}Ht'}), \end{aligned}$$

where only the spatial dependence has been specified on the rhs.

Let us multiply (8) and (9) by $:\mathrm{e}^{-\mathrm{i}a\varphi(x',t')}:$ and $:\mathrm{e}^{-\mathrm{i}(a+1)\varphi(x',t')}:$ respectively, take the trace and substitute into equation (16). We find that (16) holds due to the identity

$$\langle \mathrm{e}^{-\mathrm{i}\varphi}, \mathrm{e}^{\mathrm{i}\varphi}\bar{T}\rangle + \langle \mathbf{1}, T\rangle\langle \mathrm{e}^{-\mathrm{i}\varphi}, \mathrm{e}^{\mathrm{i}\varphi}\rangle = 2\langle \mathbf{1}, J\rangle\langle \mathrm{e}^{-\mathrm{i}\varphi}, \mathrm{e}^{\mathrm{i}\varphi}J\rangle, \tag{22}$$

where $\mathbf{1}$ is the identity operator, and we further suppressed the arguments x, x'.

To prove this identity we first write $:\mathrm{e}^{\mathrm{i}\varphi}\bar{T}:$ in terms of fermions. Using (11) we write $:\mathrm{e}^{\mathrm{i}\varphi}\bar{T} := -\lim_{y,z\to x}\nabla_y^2 : \mathrm{e}^{\mathrm{i}\varphi(x)}\,\mathrm{e}^{\mathrm{i}\varphi(z)}\,\mathrm{e}^{-\mathrm{i}\varphi(y)}:$, where we split spatial points. We now use formula (12) for vertex operators and equation (11) to write

$$:\mathrm{e}^{\mathrm{i}\varphi(x)}\bar{T}(x) := \mp\mathrm{i}(2\pi)^{3/2}\lim_{y,z\to x}\nabla_y^2\frac{(y-z)(y-x)}{(z-x)}\psi(x)\psi(z)\psi^{\dagger}(y). \tag{23}$$

This formula allows us to write the first term in (22) in terms of a correlator with four fermions insertion:

$$\mp\mathrm{i}(2\pi)^2\lim_{y,z\to x}\nabla_y^2\frac{(y-z)(y-x)}{(z-x)}\langle\psi^{\dagger}(x'), \psi(x)\psi(z)\psi^{\dagger}(y)\rangle. \tag{24}$$

Now we use a general form of the Wick theorem[9] to express the four fermion correlator in terms of correlators containing two fermions:

$$\begin{aligned}\langle\psi^{\dagger}(x'), \psi(x)\psi(z)\psi^{\dagger}(y)\rangle &= \langle\psi(x), \psi^{\dagger}(x')\rangle\langle\psi^{\dagger}(y)\psi(z), \mathbf{1}\rangle\\ &\quad-\langle\psi(z), \psi^{\dagger}(x')\rangle\langle\psi^{\dagger}(y)\psi(x), \mathbf{1}\rangle.\end{aligned} \tag{25}$$

After this, we trace back the path which led us to (24). With the help of equations (10) and (12), we obtain the formulae

$$\begin{aligned}\langle\psi^{\dagger}(x'), \psi(x)\rangle &= \frac{1}{2\pi}\langle \mathrm{e}^{-\mathrm{i}\varphi(x')}, \mathrm{e}^{\mathrm{i}\varphi(x)}\rangle,\\ \langle\mathbf{1}, \psi(z)\psi^{\dagger}(y)\rangle &= \pm\frac{1}{2\pi\mathrm{i}(y-z)}\langle\mathbf{1}, \mathrm{e}^{\mathrm{i}\varphi(z)}\,\mathrm{e}^{-\mathrm{i}\varphi(y)}\rangle\end{aligned}. \tag{26}$$

We use them in order to write the rhs of (25) in terms of bosons. After that, one applies the operation $\lim_{y,z\to x}\nabla_y^2\frac{(y-z)(y-x)}{(z-x)}$ to the rhs of (25), takes the derivative and merges the points. This leads to (22). The calculations are somewhat cumbersome, but straightforward[10].

We briefly comment on the proof of (19), the proof of the more general (18) goes along the same lines. First, we write (19) in the form

$$\langle J, J\rangle - \langle J, \mathbf{1}\rangle\langle\mathbf{1}, J\rangle = \langle \mathrm{e}^{\mathrm{i}\varphi}, \mathrm{e}^{-\mathrm{i}\varphi}\rangle\langle \mathrm{e}^{-\mathrm{i}\varphi}, \mathrm{e}^{\mathrm{i}\varphi}\rangle. \tag{27}$$

The proof of this equation is easier: after replacing bosonic exponentials and the current operator by fermionic operators according to (10), (13) one recognizes Wick's theorem.

9. Integrable hierarchy of nonlinear equations

As a final remark we mention that higher equations of the mKP hierarchy describe the evolution of Fermi gas with an arbitrary spectrum $H = \sum_p \varepsilon_p\psi_p^{\dagger}\psi_p$. They are generated by the Hirota equation described in [12]. Equations for magnetic chains equivalent to the Fermi gas can be obtained in a similar manner.

[9] See footnote 5.

[10] A technical comment on ordering mixed fermionic and bosonic strings in (24)–(26) is in order. As an example consider $\langle\psi^{\dagger}(y)\psi(x), \mathbf{1}\rangle$. A part of this correlation function is $:\mathrm{e}^{a\varphi(x)}\psi^{\dagger}(y)\psi(x):$. It is understood as $:\mathrm{e}^{a\varphi(x)}\psi^{\dagger}(y)\psi(x) := \mathrm{e}^{a\varphi_+(x)}\psi^{\dagger}(y)\psi(x)\,\mathrm{e}^{a\varphi_-(x)}$, where fermionic operators are placed in the middle between the positive and negative bosonic modes.

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Appendix A. Equation of motion for vertex operators

In this appendix we sketch the derivation of equations (8), (9) from the Riemann equation (7). First of all we show how to calculate the time derivative of the vertex operator. A subtlety here is that $\dot{\varphi}$ does not commute with φ.

We recall a simple consequence of the Hadamard lemma. If H and A are two operators such that $[[H, A], A]$ commutes with A, then

$$[H, \mathrm{e}^A] = \mathrm{e}^A \left([H, A] + \frac{1}{2!}[[H, A], A] \right) = \left([H, A] - \frac{1}{2!}[[H, A], A] \right) \mathrm{e}^A.$$

If H is a Hamiltonian $\partial_t A \sim [H, A]$ and we obtain for the time derivative of an exponent

$$\partial_t\, \mathrm{e}^A = \mathrm{e}^A \left(\dot{A} + \frac{1}{2!}[\dot{A}, A] \right) = \left(\dot{A} - \frac{1}{2!}[\dot{A}, A] \right) \mathrm{e}^A. \tag{A.1}$$

Let us now compute an evolution of $\mathrm{e}^{\mathrm{i}a\varphi}$. We assume that φ is a right chiral field. The left chiral field obeys the identical equation.

We note that the commutator $[[\dot{\varphi}^+, \varphi^+], \varphi^+]$ vanishes, and then apply (A.1). We obtain

$$\begin{aligned}
\partial_t :\mathrm{e}^{\mathrm{i}a\varphi}: &= (\partial_t\, \mathrm{e}^{\mathrm{i}a\varphi^+})\, \mathrm{e}^{\mathrm{i}a\varphi^-} + \mathrm{e}^{\mathrm{i}a\varphi^+}(\partial_t\, \mathrm{e}^{\mathrm{i}a\varphi^-}) \\
&= \mathrm{e}^{\mathrm{i}a\varphi^+} \left(\mathrm{i}a\dot{\varphi}^+ + \frac{(\mathrm{i}a)^2}{2!}[\dot{\varphi}^+, \varphi^+] \right) \mathrm{e}^{\mathrm{i}a\varphi^-} + \mathrm{e}^{\mathrm{i}a\varphi^+} \left(\mathrm{i}a\dot{\varphi}^- - \frac{(\mathrm{i}a)^2}{2!}[\dot{\varphi}^-, \varphi^-] \right) \mathrm{e}^{\mathrm{i}a\varphi^-}.
\end{aligned} \tag{A.2}$$

Let us now compute the commutator $[\dot{\varphi}^+, \varphi^+]$. Using (7) and (3) we obtain for Fourier components

$$[\dot{\varphi}_p, \varphi_q] = \frac{1}{m}(p+q)\varphi_{p+q}. \tag{A.3}$$

Then we proceed as

$$\begin{aligned}
[\dot{\varphi}^+(x), \varphi^+(x)] &= \frac{1}{m}\left(\frac{2\pi}{L}\right)^2 \sum_{p,q<0} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(p+q)x}(p+q)\varphi_{p+q} \\
&= \frac{1}{m}\left(\frac{2\pi}{L}\right)^2 \sum_{k<0} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}kx} k\varphi_p \sum_{k<p<0} 1 = -\frac{1}{\hbar m}\frac{2\pi}{L}\sum_{k<0}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}kx}k^2\varphi_k \\
&= \frac{\hbar}{m}\nabla^2\varphi^+.
\end{aligned}$$

Repeating the same calculation for φ^- we obtain

$$[\dot{\varphi}^{\pm}(x), \varphi^{\pm}(x)] = \pm\frac{\hbar}{m}\nabla^2\varphi^{\pm}. \tag{A.4}$$

Finally, using (A.4) in (A.2) we obtain (8), (9)

$$\begin{aligned}
\partial_t :\mathrm{e}^{\mathrm{i}a\varphi}: &= \mathrm{e}^{\mathrm{i}a\varphi^+} \left(\mathrm{i}a\dot{\varphi} + \frac{(\mathrm{i}a)^2}{2}\frac{\hbar}{m}\nabla^2\varphi \right) \mathrm{e}^{\mathrm{i}a\varphi^-} \\
&= \frac{\hbar}{2m}\, \mathrm{e}^{\mathrm{i}a\varphi^+}(-\mathrm{i}a : (\nabla\varphi)^2 : + (\mathrm{i}a)^2\nabla^2\varphi)\, \mathrm{e}^{\mathrm{i}a\varphi^-}.
\end{aligned} \tag{A.5}$$

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